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DYNAMIC ELASTOPLASTIC INTERACTION BETWEEN AN IMPACTOR AND A SPHERICAL SHELL

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Dynamic axisymmetric elastoplastic interaction between a massive body and a simply supported, circular segment of a spherical shell is studied. The problem of determining the contact-interaction force is formulated for the case of spherical and conical bodies. A nonlinear integral equation is derived for various models of local plastic compression using the equations of equilibrium of a membrane spherical shell written in terms of radial displacement of the shell. Numerical results are presented graphically.

Assuming that the mutual velocity is much lower than the velocity of elastic waves in the materials, we reduce the dynamic problem to a quasistatic problem by ignoring inertia effects in the local-compression zone. Displacements of the shell are considered elastic except in the contact zone, where elastoplastic deformation occurs. Initially, the shell is at rest. A body of mass m with elastic constants E_2 and ν_2 and plastic constant k_2 impacts on the shell vertex.

We denote the displacement of the falling body by s, the displacement of the shell at the contact point by w, and the local plastic compression by α. In this case, we have [1]

$$s = w + \alpha. \tag{1}$$

To determine the displacement of the impactor s, we use the differential equation of motion $m\ddot{s} = -P(t)$. Integration of this equation subject to the initial conditions $s_0 = 0$ and $\dot{s}_0 = V_0$ yields

$$s(t) = V_0 t - \frac{1}{m}\int_0^t \int_0^{t_1} P(t_2)\, dt_2\, dt_1, \tag{2}$$

where V_0 is the initial velocity of the impactor directed along the shell radius.

The displacement of the shell due to the force applied to its vertex is determined from the equations of motion of a membrane spherical shell:

$$(N_\varphi \sin\varphi)_{,\varphi} - N_\theta \cos\varphi = \rho h R_1 \ddot{u}_\varphi \sin\varphi, \qquad N_\varphi + N_\theta = -\rho h R_1 \ddot{w} + q_3 R_1; \tag{3}$$

$$\begin{gathered} N_\varphi = E_1 h((1-\nu_1^2)R_1)^{-1}(u_{\varphi,\varphi} + w + \nu_1(u_\varphi \cot\varphi + w)), \\ N_\theta = E_1 h((1-\nu_1^2)R_1)^{-1}(u_\varphi \cot\varphi + w + \nu_1(u_{\varphi,\varphi} + w)). \end{gathered} \tag{4}$$

Here ρ is the density of the material, h and R_1 are the thickness and radius of the shell, q_3 is the load, and E_1 and ν_1 are the elastic constants of the shell; the coordinate lines φ and θ are directed along a meridian and a parallel, respectively. The plastic constant of the shell is denoted by k_1.

The boundary conditions have the form

$$u_\varphi|_{\varphi=\varphi_0} = 0, \qquad w|_{\varphi=\varphi_0} = 0, \tag{5}$$

where φ_0 is the shell opening angle.

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We introduce the following dimensionless quantities: $v = u_\varphi/R_1$, $w = w/R_1$, $\tau = tc/R_1$, and $c^2 = E_1((1-\nu_1^2)\rho)^{-1}$. Then, Eqs. (3) and (4) are written as

$$(N_\varphi \sin\varphi)_{,\varphi} - N_\theta \cos\varphi = E_1 h(1-\nu_1^2)^{-1} v_{,\tau\tau} \sin\varphi,$$

$$N_\varphi + N_\theta = -E_1 h(1-\nu_1^2)^{-1} w_{,\tau\tau} + q_3 R_1,$$

$$N_\varphi = E_1 h(1-\nu_1^2)^{-1}(v_{,\varphi} + w + \nu_1(v\cot\,\varphi + w)),$$

$$N_\theta = E_1 h(1-\nu_1^2)^{-1}(v\cot\,\varphi + w + \nu_1(v_{,\varphi} + w)).$$

Elimination of the forces N_φ and N_θ from these equations yields

$$v_{,\varphi\varphi}\sin\varphi + v_{,\varphi}\cos\varphi - (\cot\,\varphi\cos\varphi + \nu_1\sin\varphi)v + (1+\nu_1)w_{,\varphi}\sin\varphi = v_{,\tau\tau}\sin\varphi,$$

$$(1+\nu_1)(v_{,\varphi} + v\cot\,\varphi + 2w) = -w_{,\tau\tau} + q, \qquad q = (1-\nu_1^2)(E_1 h)^{-1} R_1 q_3.$$

We make the replacement $v_\varphi = v\sin\varphi$:

$$v_{\varphi,\varphi\varphi} - v_{\varphi,\varphi}\cot\,\varphi + (1-\nu_1)v_\varphi + (1+\nu_1)w_{,\varphi}\sin\varphi = v_{\varphi,\tau\tau},$$

$$(1+\nu_1)(v_{\varphi,\varphi}\sin^{-1}\varphi + 2w) = -w_{,\tau\tau} + q.$$

We apply the Laplace transform over time t denoting the images v_φ, w, and q by v_φ^*, w^*, and q^*, respectively:

$$\begin{gathered} v^*_{\varphi,\varphi\varphi} - v^*_{\varphi,\varphi}\cot\,\varphi + (1-\nu_1-p^2)v^*_\varphi + (1+\nu_1)w^*_{,\varphi}\sin\varphi = 0, \\ (1+\nu_1)v^*_{\varphi,\varphi}\sin^{-1}\varphi + (2(1+\nu_1)+p^2)w^* = q^*. \end{gathered} \tag{6}$$

System (6) can be written as

$$(v^*_{\varphi,\varphi}\sin^{-1}\varphi)_{,\varphi} + (1-\nu_1-p^2)v^*_\varphi\sin^{-1}\varphi + (1+\nu_1)w^*_{,\varphi} = 0,$$

$$(1+\nu_1)(v^*_{\varphi,\varphi}\sin^{-1}\varphi)_{,\varphi} + (2(1+\nu_1)+p^2)w^*_{,\varphi} = q^*_{,\varphi}.$$

Eliminating $(v^*_{\varphi,\varphi}\sin^{-1}\varphi)_{,\varphi}$, we obtain

$$v^*_\varphi = \sin\varphi((1+\nu_1)(1-\nu_1-p^2))^{-1}(w^*_{,\varphi}(1-\nu_1^2+p^2) - q^*_{,\varphi}).$$

Differentiation of the expression for v^*_φ with respect to φ yields

$$v^*_{\varphi,\varphi} = ((1+\nu_1)(1-\nu_1-p^2))^{-1}((1-\nu_1^2+p^2)(w^*_{,\varphi\varphi}\sin\varphi + w^*_{,\varphi}\cos\varphi) - (q^*_{,\varphi\varphi}\sin\varphi + q^*_{,\varphi}\cos\varphi)).$$

Inserting the last expression into the second equation in (6), we obtain the following equation for w^*:

$$\nabla^2 w^* + (2(1-\nu_1^2) - (1+3\nu_1)p^2 - p^4)(1-\nu_1^2+p^2)^{-1}w^* = (1-\nu_1^2+p^2)^{-1}(\nabla^2 q^* + (1-\nu_1-p^2)q^*), \tag{7}$$

where $\nabla^2 = \partial^2/\partial\varphi^2 + (\partial/\partial\varphi)\cot\,\varphi$.

We seek a solution of (7) in the form of a series in Legendre polynomials which possess completeness and satisfy the boundary conditions (5):

$$w^* = \sum_{n=0}^{\infty} w_n^* P_n(\cos(\delta_1\varphi)), \qquad \delta_1 = \frac{\pi}{2\varphi_0}.$$

The point load $q(t,\varphi) = P(t)\delta(\varphi)$ is also expanded in a series in Legendre polynomials:

$$q = P(t)(2\pi R_1^2(1-\cos\varphi_0))^{-1}\sum_{n=0}^{\infty}(2n+1)P_n(\cos(\delta_1\varphi)),$$

$$q^* = P^*(p)(2\pi R_1^2(1-\cos\varphi_0))^{-1}\sum_{n=0}^{\infty}(2n+1)P_n(\cos(\delta_1\varphi)).$$

Substituting the expansions of w^* and q^* into (7), we obtain

$$w_n^* = P^*(1-\nu_1^2)(2n+1)(p^2+B)(2\pi R_1 h E_1(1-\cos\varphi_0)(p^4+p^2A_2+A_0))^{-1},$$

$$B = n\delta_1(n\delta_1+1)+\nu_1-1, \qquad A_2 = n\delta_1(n\delta_1+1)+3\nu_1+1,$$

$$A_0 = (1-\nu_1^2)n\delta_1(n\delta_1+1)-2(1-\nu_1^2).$$

Since

$$p^4+p^2A_2+A_0 = (p^2+\omega_1^2)(p^2+\omega_2^2), \qquad \omega_1^2 = (A_2+\sqrt{A_2^2-4A_0})/2, \qquad \omega_2^2 = (A_2-\sqrt{A_2^2-4A_0})/2,$$

we obtain an expression for which the Laplace transform is tabulated [2]. Finally, the displacement of the shell w takes the form

$$w(\varphi,\tau) = \frac{1-\nu_1^2}{2\pi R_1 h E_1(1-\cos\varphi_0)} \int\limits_0^\tau P(\tau_1) \sum_{n=0}^{\infty}(2n+1)[L_{1n}\sin(\omega_1(\tau-\tau_1))+L_{2n}\sin(\omega_2(\tau-\tau_1))]P_n(\cos(\delta_1\varphi))\,d\tau_1, \tag{8}$$

$$L_{in} = (B-\omega_i^2)(\omega_i(\omega_1^2-\omega_2^2))^{-1}, \qquad i=1,2.$$

Substituting (2) and (8) and expressions for α corresponding to various models of local plastic compression into (1), we arrive at a nonlinear integral equation for $P(t)$. This equation is solved by the following iterative scheme [1]:

1) $\tau_i = \tau i$;
2) $s_i = s_{i-1}+V_{i-1}\tau+y_{i-1}\tau^2/2$;
3) $w_i = \tau\,\dfrac{1-\nu_1^2}{2\pi R_1 h E_1(1-\cos\varphi_0)}\displaystyle\sum_{n=0}^{\infty}\sum_{j=1}^{i-1}P_j(2n+1)[L_{1n}\sin(\omega_1(i-j)\tau)+L_{2n}\sin(\omega_2(i-j)\tau)]$;
4) $\alpha_i = s_i - w_i$;
5) P_i is calculated for α_i;
6) $y_i = -P_i/m$;
7) $V_i = V_{i-1}+y_i\tau$.

The initial conditions are $V|_{i=0} = V_0/c$, $s_0 = 0$, and $y_0 = 0$.

Given α_i, we calculate P_i using the solution of the contact problem. The following models are employed:

— For a spherical impactor with curvature radius at the contact point R_2:

1) elastoplastic model [3, 4]

$$\alpha = \begin{cases} bP^{2/3}, & P_{\max} < P_1, \quad dP/dt > 0, \\ b_f P^{2/3} + \alpha_p(P_{\max}), & dP/dt < 0, \quad P_{\max} > P_1, \\ (1+\beta)c_1P^{1/2}+(1-\beta)Pd, & dP/dt > 0, \quad P_{\max} > P_1, \end{cases} \tag{9}$$

where $b = R^{-1/3}(3/(4E))^{2/3}$, $R^{-1} = R_2^{-1} - R_1^{-1}$, $E = E_1E_2((1-\nu_1^2)E_2+(1-\nu_2^2)E_1)^{-1}$, $P_1 = \chi^3(3R/(4E))^2$, $\chi = \pi k\lambda$ (k is the smallest of the two plastic constants of the colliding bodies and $\lambda = 5.7$), $b_f = R_f^{-1/3}(3/(4E))^{2/3}$, $R_f = (4/3)EP_{\max}^{1/2}\chi^{-3/2}$, $\alpha_p(P_{\max}) = (1-\beta)P_{\max}(2\chi R_p)^{-1}$, $R_p^{-1} = R^{-1} - R_f^{-1}$, $\beta = 0.33$, $c_1 = 3\chi^{1/2}(8E)^{-1}$, and $d = (2\chi R)^{-1}$;

2) Kil'chevskii model [5]

$$\alpha = \begin{cases} bP^{2/3}, & P < P_0, \quad dP/dt > 0, \\ bP^{2/3}+Pd, & P > P_0, \quad dP/dt > 0, \\ bP^{2/3}+P_{\max}d, & P_{\max} > P_0, \quad dP/dt < 0, \end{cases} \tag{10}$$

where $P_0 = (4/3)Ea_0^3R^{-1}$ and $a_0 = \pi kR(0.62E)^{-1}$;

3) Hertz model

$$\alpha = bP^{2/3}; \tag{11}$$

4) rigid-plastic model [follows from (9) if the elastic terms are ignored]

$$\alpha = (1-\beta)Pd; \tag{12}$$

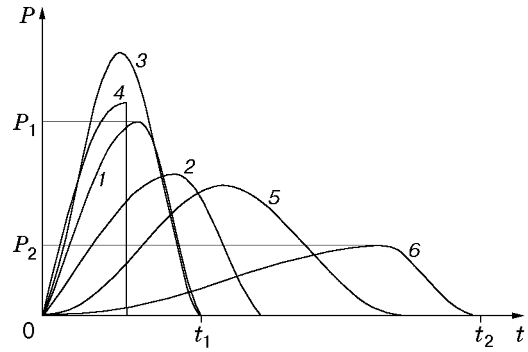

Fig. 1. Dependence $P(t)$ for $V_0 = 0.5$ m/sec ($P_1 = 4470.14$ N, $P_2 = 1609.58$ N, $2\gamma = 150^\circ$, $t_1 = 0.0002531647$ sec, and $t_2 = 0.0006991468$ sec): curve 1 refers to the elastoplastic model for a sphere (9), curve 2 to the Kil'chevskii model (10), curve 3 to the Hertz model (11), curve 4 to the rigid-plastic model (12), curve 5 to the elastic model for a cone (13), and curve 6 to the elastoplastic model for a cone (14).

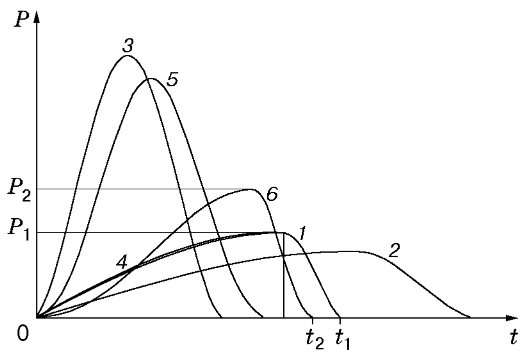

Fig. 2

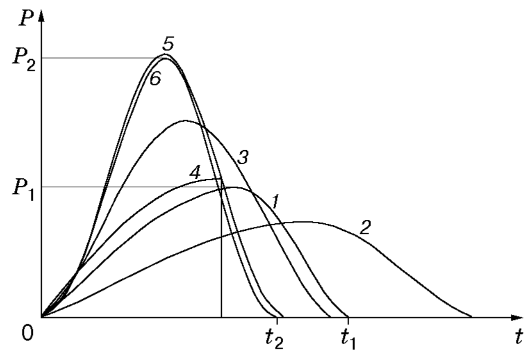

Fig. 3

Fig. 2. Dependence $P(t)$ for $V_0 = 50$ m/sec ($P_1 = 490.28$ kN, $P_2 = 744.3$ kN, $2\gamma = 150^\circ$, $t_1 = 0.0001648514$ sec, and $t_2 = 0.0001501326$ sec) (notation same as in Fig. 1).

Fig. 3. Dependence $P(t)$ for $V_0 = 1$ m/sec ($P_1 = 9194.98$ N, $P_2 = 18\,352.5$ N, $2\gamma = 178^\circ$, $t_1 = 0.0002296145$ sec, and $t_2 = 0.0001766265$ sec) (notation same as in Fig. 1).

— For a conical impactor with the opening angle 2γ:
1) elastic model [6]

$$\alpha = (\pi \cot \gamma/(2E))^{1/2} P^{1/2}; \tag{13}$$

2) elastoplastic model [7]

$$\alpha = \begin{cases} c_2 P^{1/2}, & dP/dt > 0, \\ (P\chi)^{1/2} E_1^{-1} + \alpha_{p,\max}, & dP/dt \leqslant 0, \end{cases} \tag{14}$$

where $c_2 = \cot \gamma(1-\delta)\chi^{-1/2} + (1 + 2(\delta - 1)/\pi)\chi^{1/2}E^{-1}$, $\alpha_{p,\max} = (1-\delta)(P_{\max}/\chi)^{1/2}(\cot \gamma - 2\chi/(\pi E))$, and $\delta = 0.22$.

Figures 1–3 show curves of $P(t)$ obtained with the use of the models of local plastic compression (9)–(14) for the following parameters: shell radius $R_1 = 1$ m, shell thickness $h = 0.01$ m, shell opening angle $\varphi_0 = 90^\circ$, radius of the spherical impactor $R_2 = 0.02$ m, and mass of the impactor $m = 0.25$ kg. The shell and impactor were made of steel [P_1, P_2 and t_1, t_2 are the maximum values of the contact force and the duration of contact for spherical and conical impactors for the elastoplastic models (9) and (14), respectively].

One can see from Figs. 1–3 that the solutions based on models (9) and (14) agree well with the experimental data of [8]. The Hertz model (11) gives satisfactory results for $V_0 < 0.15$ m/sec, and the rigid-plastic model (12) is applicable only for $V_0 > 10$ m/sec. For the elastic model of a cone (13), the error in determining the main characteristics of the impact can be as great as 100%. The Kil'chevskii model (10) also leads to a considerable error.

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